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HYDRODYNAMICS OF A POINT VORTEX RING

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The kinematics and dynamics of a point vortex ring in an incompressible fluid and its interaction with a surface are considered.

In [1], a circular vortex with a zero fluid velocity on its axis is studied. The axis of the vortex is a circle of radius R, around which the fluid motion occurs. In the present article this circle is assumed to be a closed vortex filament with infinite circulation. In this case, when any other fluid flow is superposed on the circular vortex, its axis, unlike [1], does not change its position in space, thus allowing extension of the range of imposed flows.

A solution of the problem is obtained in a toroidal coordinate system (σ, τ, φ) [2].

Kinematics of Vortex Ring. The statement of the problem of a toroidal vortex is given in [1]. In the case of a point vortex the boundary conditions on the vortex axis have the form

$$V_\sigma \to \infty \quad \text{when} \quad \tau \to \infty . \tag{1}$$

By integrating the continuity equation, we obtain the following expression for the velocity:

$$V_\sigma = c(\tau)(\text{ch}\,\tau - \cos\sigma)^2 . \tag{2}$$

Let us consider a particular case that satisfies condition (1): $c = c(\tau) = c_1 = \text{const}$. According to the boundary condition at the symmetry point of the torus

$$V_\sigma = V_0 = \text{const} \quad \text{when} \quad \sigma = \pm\pi , \quad \tau \to 0 ,$$

from Eq. (2) we obtain $c_1 = V_0/4$. We now write expression (2) in cylindrical coordinates (z, y, φ) [1, 2]:

$$V_\sigma = \frac{V_0 a^4}{((y-a)^2 + z^2)((y+a)^2 + z^2)} .$$

In projection onto the axis of the coordinates we obtain:

$$U_z = \frac{V_0 a^4 (y^2 - a^2 - z^2)}{[((y-a)^2 + z^2)((y+a)^2 + z^2)]^{3/2}} , \tag{3}$$

$$U_y = -\frac{V_0 a^4 2yz}{[((y-a)^2 + z^2)((y+a)^2 + z^2)]^{3/2}} , \tag{4}$$

hence, upon integration of the equations $U_y = -(1/y)\cdot(\partial\Psi_0/\partial z)$ and $U_z = (1/y)\cdot(\partial\Psi_0/dy)$, we write the expression for the stream function

$$\Psi_0 = -\frac{V_0 a^2 (z^2 + y^2 + a^2)}{4\sqrt{((y-a)^2 + z^2)((y+a)^2 + z^2)}} . \tag{5}$$

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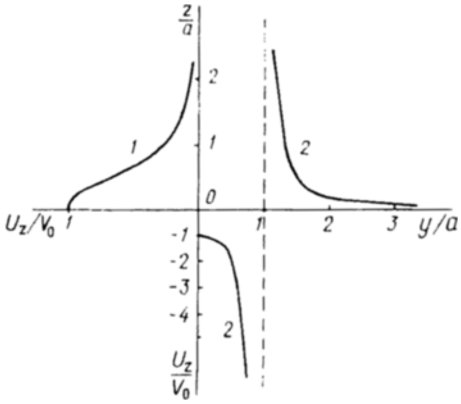

Fig. 1. Variation of velocity U_z/V_0 along the coordinate axes: 1) along z/a axis; 2) along the y/a axis.

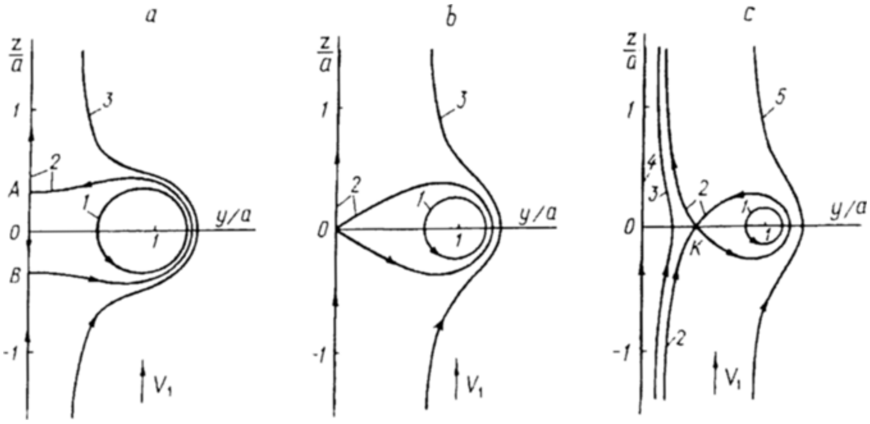

Fig. 2. Distribution of streamlines Ψ_* in flow around vortex ring; a) $k = 0.8$: 1) $\Psi_* = -0.325$; 2 -0.25; 3 -0.2; b) $k = 1$: 1) $\Psi_* = -0.5$; 2) -0.25; 3) -0; c) $k = 1.5$: 1) $\Psi_* = -1$; 2) -0.2247; 3) -0.2375; 4) -0.25; 5) 0.325.

Vortex Ring in Uniform Rectilinear Flow. As a result of superposition of the stream functions of a toroidal vortex Ψ_0 and of a rectilinear flow $\Psi_1 = V_1 y^2/2$ [1], we have $\Psi = \Psi_0 + \Psi_1$ or in dimensionless form (using the notation $z = z/a$, $y = y/a$),

$$\Psi_* = \frac{\Psi}{V_0 a^2} = k\frac{y^2}{2} - \frac{z^2 + y^2 + 1}{4\sqrt{((y-a)^2 + z^2)((y+a)^2 + z^2)}},$$

where $k = V_1/V_0$. Then we express the function $z = z(y, k, \Psi_*)$ and obtain an equation for the trajectories of the fluid particles

$$z = \left[\frac{2y}{\sqrt{1 - (2ky^2 - 4\Psi_*)^{-2}}} - y^2 - 1\right]^{1/2}. \tag{6}$$

Stationary Vortex in the Vicinity of an Impermeable Surface Perpendicular to the Vortex Symmetry Axis. Suppose a surface lies in the xOy plane. We consider two vortices with the same directions of circulation located as follows:

1) the axis of the first torus lies in the $z = h$ plane;

2) the axis of the second torus lies in the $z = -h$ plane,

which are described by the stream functions Ψ_1 and Ψ_2, velocities U_{z1}, U_{z2}, U_{y1}, and U_{y2}, similarly to formulas (3)-(5). Only for the first torus should z be replaced by $z - h$, and for the second torus, by $z + h$. The addition of these two circular vortices at $z = 0$ yields

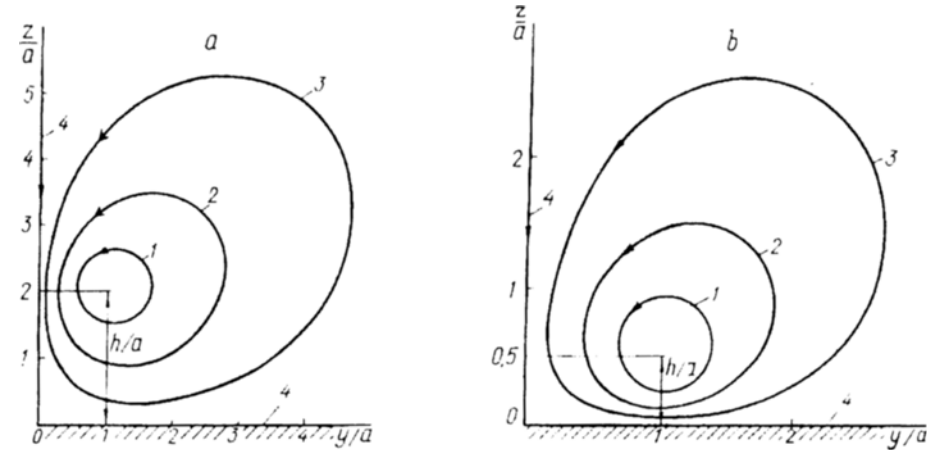

Fig. 3. Distribution of streamlines Ψ_* near a plane: 1) $\Psi_* = -0.25$; 2) -0.05; 3) -0.075; 4) -0; a) $h/a = 2$; b) 0.5.

$$U_{y12} = U_{y1} + U_{y2} = 0\,, \tag{7}$$

$$U_{z12} = U_{z1} + U_{z2} = \frac{2V_0a^4(y^2 - a^2 - h^2)}{[((y-a)^2 + h^2)((y+a)^2 + h^2)]^{3/2}}\,.$$

Thus, in order to obtain an impermeable surface, it is necessary to add a third circular vortex with the axis located in the $z = 0$ plane and with the velocity profile

$$U_{y3} = 0\,, \quad U_{z3} = -U_{z12} \quad \text{at} \quad z = 0\,. \tag{8}$$

Proceeding from Eq. (7), we assume that

$$U_{z3} = -2V_0a^4 \frac{(y^2 - a^2 - h^2 - z^2)}{[((y-a)^2 + h^2 + z^2)((y+a)^2 + h^2 + z^2)]^{3/2}}\,,$$

where $U_{z3} = (1/y)\cdot(\partial\Psi_3/\partial y)$. Then, integrating this expression and taking into account the formula for U_{z3}, we have

$$\Psi_3 = V_0a^2 \frac{y^2 + h^2 + a^2 + z^2}{2\sqrt{((y-a)^2 + h^2 + z^2)((y+a)^2 + h^2 + z^2)}}\,,$$

$$U_{y3} = \frac{4V_0a^4zy}{[((y-a)^2 + h^2 + z^2)((y+a)^2 + h^2 + z^2)]^{3/2}}\,.$$

We found that a stationary point vortex located near the surface at distance h is described by the following expressions:

$$\Psi = \Psi_1 + \Psi_2 + \Psi_3\,, \quad U_z = U_{z1} + U_{z2} + U_{z3}\,, \quad U_y = U_{y1} + U_{y2} + U_{y3}\,. \tag{9}$$

Results of Calculations. Figure 1 presents the results of calculation of the velocity profiles of a stationary vortex in space. (Curve 1 represents the velocity profile on the symmetry axis of the torus ($y = 0$).) This velocity profile coincides with the results of [1] for a vortex with a zero velocity on the torus axis. Curves 2 represent the distribution of velocities at $z = 0$. The calculations were performed by formula (3).

The results of the interaction of a circular vortex with a rectilinear flow are presented in Fig. 2 in the form of streamlines depending on the coefficient $k = V_1/V_0$. From Figs. 2a and 2b ($k \leq 1$) it follows that the flow has

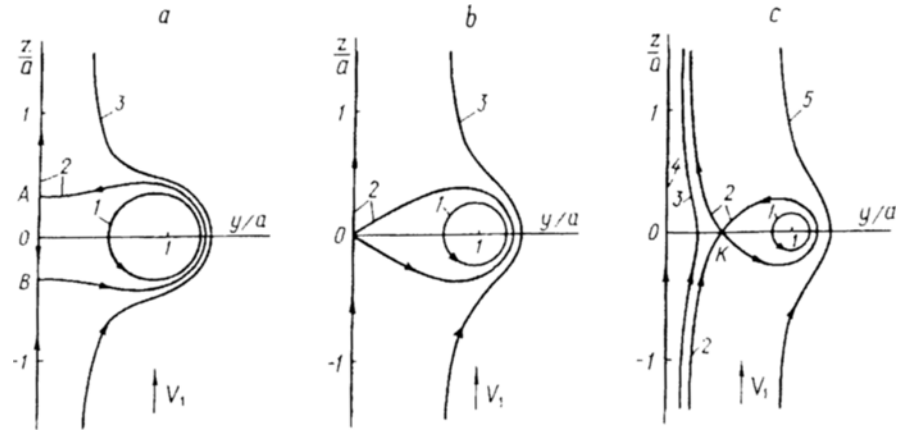

Fig. 4. Variation of shear stresses on a plane along the y/a axis: 1) $h/a = 0.5$; 2) 1.

two regions: an internal vortex region and a region of flow around the toroidal vortex. As the coefficient k increases, the critical points of the flow A and B come closer together and at $V_1 = V_0$ ($k = 1$) merge into one point 0. A further increase in the coefficient k leads to the appearance of a third flow region: a fluid flow moving inside the circular vortex along the axis of its symmetry. This is shown in Fig. 2c, where K is the critical point ($V_K = 0$). Thus, here, unlike [1], the flow velocity at infinity $U_\infty = V_1$ is not limited by the values $0 < V_1 < V_0$, i.e., in principle, we can consider a circular vortex moving with any velocity. The streamlines were calculated by formula (6).

The results of calculation by the first expression of system (9) for the streamlines of a stationary vortex near a solid surface at different distances from it are presented in Fig. 3.

Now, we estimate the distribution of the shear stress on a plane with liquid flow formed by a circular vortex located in the vicinity of the plane. According to the Newton law, we have

$$\tau = \mu \frac{\partial U_y}{\partial z} \quad \text{at} \quad z = 0 .$$

The results of calculations in the form of the dependence of the dimensionless stresses $\tau a/\mu V_0$ on the radial coordinate y/a are given in Fig. 4. The dependence corresponding to the ratio $h/a = 2$ is not shown, because the maximum value of shear stresses at this ratio is equal to 0.1. It is seen from the figure that as the circular vortex approaches the plane, the shear stresses increase sharply. Thus, when the ratio h/a decreases by a factor of two, the value of $\tau a/\mu V_0$ increases by a factor of about ten. For circular vortices corresponding to Figs. 3a and 3b the maximum shear stresses on the plane are equal to 11.6 and 1.3, respectively.

NOTATION

σ, τ, φ, toroidal coordinates; $V(V_\sigma, V_\tau, V_\varphi)$, velocity of fluid particle and its projection in toroidal coordinates; V_0, velocity at center of vortex ring on the axis of its symmetry; z, y, φ, cylindrical coordinates; a, distance from the torus axis to the axis of its symmetry (0z); U_z, U_y, velocities in cylindrical coordinate system; Ψ_0, stream function of stationary vortex ring; $U_\infty = V_1$, velocity of rectilinear flow at infinity; Ψ_1, stream function of rectilinear flow; $\Psi = \Psi_0 + \Psi_1$, superposition of two flows; $k = V_1/V_0$, coefficient of the velocity ratio; $\Psi_* = \Psi/V_0 a^2$, dimensionless stream function; h, distance from axis of circular vortex to the x0y plane; τ, shear stress; μ, coefficient of dynamic viscosity.

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